SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black pen.
- Board approved calculators may be used
- Show all necessary working in Questions 11–16
- A table of standard integrals is on the back of the multiple choice answer sheet

Total Marks - 100 Marks

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **90 Marks**

- Attempt Questions 11–16
- Allow about 2 hour 45 minutes for this section.

Examiner: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

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Section I **Objective-response Questions**
Total marks – 10
Attempt Questions 1 – 10

Answer each question on the multiple choice answer sheet provided.

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1 Let $u = 7\cos\frac{\pi}{4} + 7i\sin\frac{\pi}{4}$ and $v = a\cos b + ai\sin b$, where a and b are real constants.

If $uv = 42\cos\frac{\pi}{20} + 42i\sin\frac{\pi}{20}$, then

(A) $a = 35$ and $b = \frac{\pi}{5}$ (B) $a = 6$ and $b = \frac{\pi}{5}$

(C) $a = 35$ and $b = -\frac{\pi}{5}$ (D) $a = 6$ and $b = -\frac{\pi}{5}$

2 If $z^2 = 4\text{cis}\left(\frac{4\pi}{3}\right)$, then z is equal to

(A) $\sqrt{3}+i$ or $-\sqrt{3}-i$ (B) $1-\sqrt{3}i$ or $-1+\sqrt{3}i$

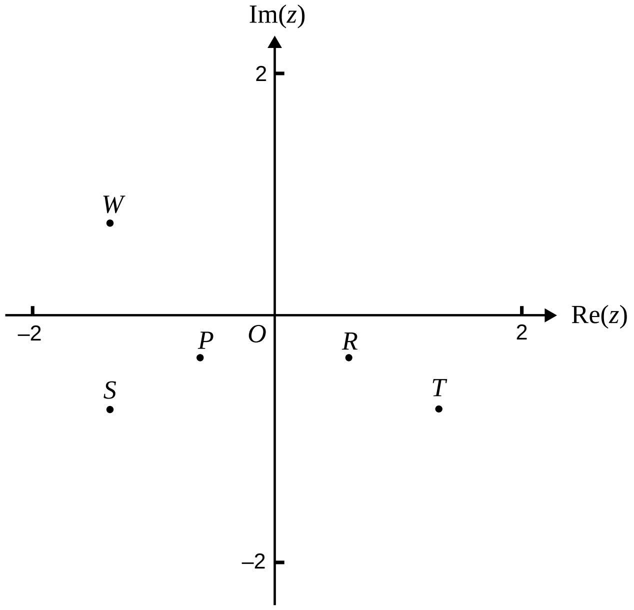
(C) $\sqrt{3}-i$ or $\sqrt{3}+i$ (D) $1-\sqrt{3}i$ or $1+\sqrt{3}i$

3 Let $z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$.
The imaginary part of $z - i$ is

(A) $-\frac{i}{2}$ (B) $-\frac{3i}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

4 The point W on the Argand diagram below represents a number w where $|w| = 1\cdot5$.
The number w^{-1} is best represented by the point

(A) P (B) R (C) S (D) T



5 $P(z)$ is a polynomial in z of degree 4 with real coefficients
Which one of the following statements must be false?

(A) $P(z)$ has four real roots.
(B) $P(z)$ has two real roots and two non-real roots.
(C) $P(z)$ has one real root and three non-real roots.
(D) $P(z)$ has no real roots.

6 The graph of $f(x)=\dfrac{1}{x^2+mx-n}$, where m and n are real constants, has no vertical asymptotes if

(A) $m^2<-4n$ (B) $m^2>-4n$ (C) $m^2<4n$ (D) $m^2>4n$

7 Consider the graph of $f(x)=\sin^3 x$ for $-\pi\le x\le 2\pi$.
The area bounded by the graph of $f(x)$ and the x-axis could be found by evaluating

(A) $\int_{-1}^{1}(1-u^2)\,du$ (B) $3\int_{-1}^{1}(1-u^2)\,du$

(C) $-\int_{-1}^{1}(1-u^2)\,du$ (D) $-3\int_{-1}^{1}(1-u^2)\,du$

8 Given that $\dfrac{dy}{dx}=y^2+1$, and that $y = 1$ at $x = 0$, then

(A) $y=y^2x+x+1$ (B) $y=\tan\left(x+\dfrac{\pi}{4}\right)$

(C) $y=\tan\left(x-\dfrac{\pi}{4}\right)$ (D) $x=\log_e\left(\dfrac{y^2+1}{2}\right)$

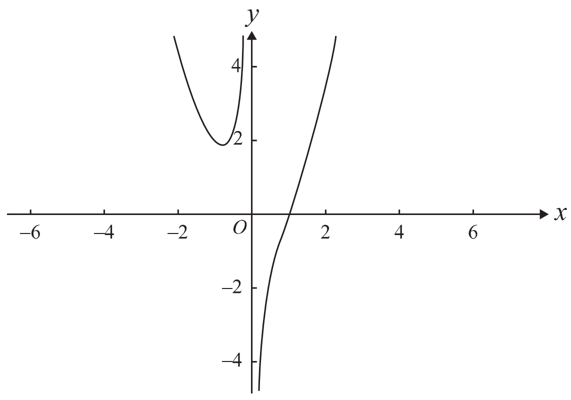
9 The velocity v m/s of a body which is moving in a straight line, when it is x m from the origin, is given by $v=\sin^{-1}x$.
The acceleration of the body in m/s^2 is given by

(A) $-\cos^{-1}x$ (B) $\cos^{-1}x$ (C) $-\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$ (D) $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

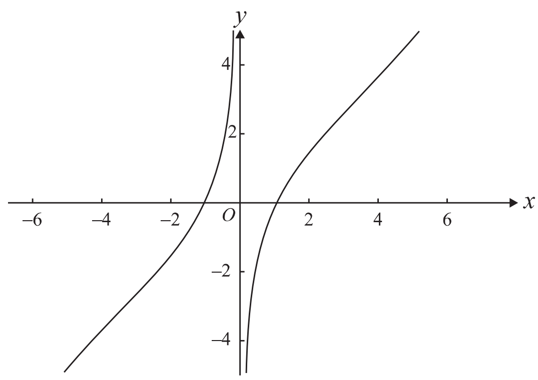
10 Let $f(x) = \dfrac{x^k + a}{x}$, where k and a are real constants.

If k is an odd integer which is greater than 1 and $a < 0$, a possible graph of f could be

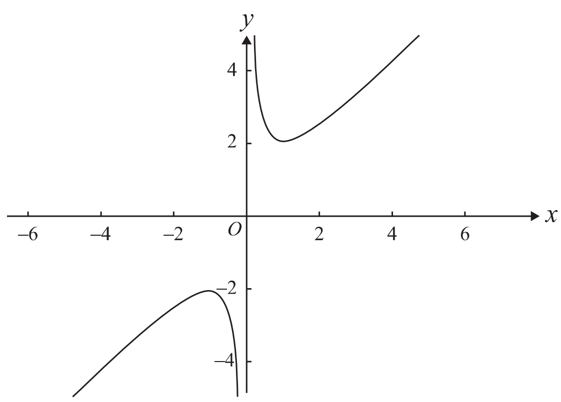
(A)



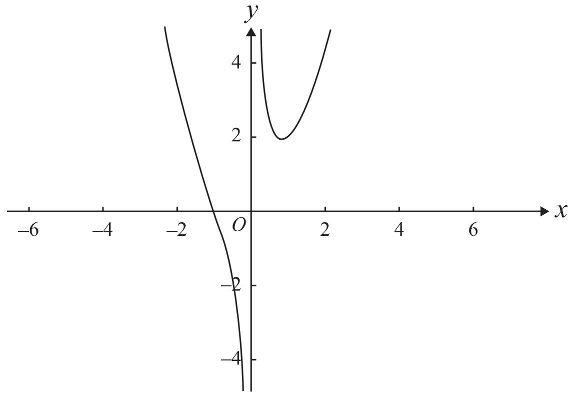
(B)



(C)



(D)



End of Section I

Section II **Free response Questions**
Total marks – 90
Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $x = \sin^2\theta$ to evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}}\,dx$ **3**

(b) Find $\displaystyle\int x\sqrt{3-x}\,dx$. **2**

(c) (i) By completing the square, find the exact value of $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{\sqrt{2x(1-2x)}}\,dx$ **2**

(ii) Hence, evaluate $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1-x}{\sqrt{2x(1-2x)}}\,dx$ **2**

(d) Find the value of the discriminant for the quadratic equation **2**

$$(1+i)z^2 + 4iz - 2(1-i) = 0$$

(e) (i) Find the value of $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6$. **1**

(ii) Show that $(\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta) = 1+\cos\theta + i\sin\theta$. **1**

(iii) Hence show that $\left(1+\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 + \left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6 = 0$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The line $x = 8$ is a directrix of the ellipse with equation **2**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a>b>0$$

and (2, 0) is the corresponding focus.
Find the value of a and b.

(b) (i) Show that $2-i$ is a solution of the equation $z^3-(2-i)z^2+z-2+i=0$. **2**

(ii) Hence find all the solutions of the equation $z^3-(2-i)z^2+z-2+i=0$. **2**

(c) Consider the function $f(x)=\log_e\left(4-x^2\right)$.

(i) By first sketching $y=4-x^2$, sketch $y=f(x)$. **2**

Let A be the magnitude of the area enclosed by the graph of $y=f(x)$, the coordinate axes and the line $x = 1$.

(ii) Without evaluating A, use (i) to show that $\log_e 3 < A < \log_e 4$. **1**

(iii) Find $\displaystyle\int \frac{x^2}{4-x^2}\,dx$. **3**

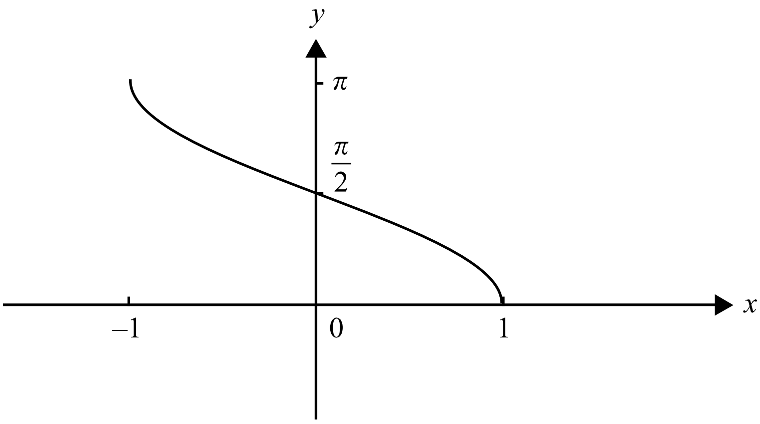
(iv) Hence find the exact value of A in the form $a+b\log_e c$, where a, b and c are integers. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove using induction for integers $n \geq 2$. **3**

$$n^{n+1} > n(n+1)^{n-1}$$

(b) The diagram below shows the graph of $y = \cos^{-1} x$.



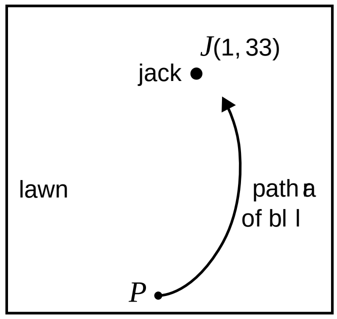
Using the method of cylindrical shells, find the exact volume formed if the graph above is rotated about the y-axis. **3**

Question 13 continues on the next page

Question 13 continued

(c) The game of lawn bowls is played on a horizontal lawn.
The aim is to roll a ball (usually called a ‘bowl’) to come to rest as close as possible to a target ball called the ‘jack’.

Bowler



View from above

All displacements are in metres.
At one stage during the game, the jack is at the point $J(1,33)$.
The path of a particular ball in this game is modelled by:

$$x = 2\sin\left(\frac{2t}{15}\right) \text{ and } y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right),\ 0 \le t \le \frac{15\pi}{2}$$

where t is the time in seconds after the ball is released from the point P.

(i) Write down the coordinates of P. **1**

(ii) Find expressions for the components of velocity, in metres per second, of the ball at time t seconds after the ball is released. **2**

(iii) At the instant the ball is released, what angle does its path make with the forward direction?
Give your answer correct to 1 decimal place. **2**

(iv) At what time, correct to the nearest tenth of a second, does the ball begin to swing left towards the jack? **2**

(v) Determine whether the path of the ball passes through J. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

A ‘parasailing’ water-skier i.e. a water-skier with a parachute attached of mass 90 kg is towed by a boat in a straight line from rest.
The boat exerts a constant force of 410 N acting horizontally on the skier.
At this stage the resistance acting on the skier is a constant 50 N, which acts horizontally.

(a) By use of a force diagram, show that the acceleration of the skier is 4 m/s^2. **2**

(b) By starting with $a = 4$, show that the speed of the skier, is given by $v^2 = 8x$, where x is the horizontal distance travelled by the skier. **2**
Hence show that having been towed a distance of 32 m, his speed is 16 m/s.

After the skier has been towed 32 m across the water the drag of the parachute becomes significant. The drag of the parachute produces an *additional* resistance of $6v$ N to the horizontal motion of the skier, where v m/s is the velocity of the skier.
Let a m/s^2 is the acceleration of the skier.

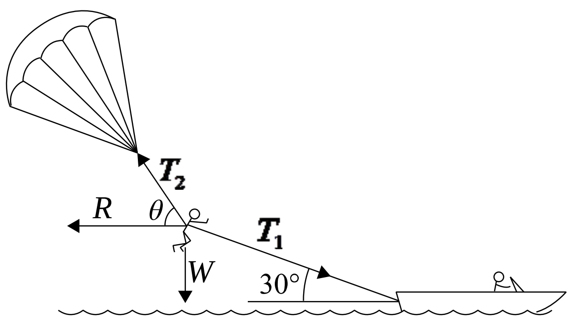
(c) Show that $a = \dfrac{1}{15}(60 - v)$ **1**

(d) Find the time required to reach a speed of 20 m/s from a speed of 16 m/s. **3**
Give your answer in seconds, correct to one decimal place.

After some time, the parasailing skier is being towed horizontally at a *constant speed* and at a fixed distance above the water.

The tow rope from the boat makes an angle of $30°$ to the horizontal, and the parachute cord makes an angle of θ to the horizontal.

The diagram below shows all the forces that are now acting on the parasailing water skier:
The tow rope now exerts a force, T_1, of 500 N on the skier.
The skier is experiencing a horizontal resistance, R, of 100 N.
Let the tension exerted by the parachute cord on the skier be T_2, and the force due to gravity on the skier be W.
Take $g = 10$, where g is the magnitude of the acceleration due to gravity.



(e) By resolving in the horizontal and vertical directions, show that **3**

$$\begin{cases} 500\cos 30° - T_2\cos\theta - 100 = 0 \\ T_2\sin\theta - 500\sin 30° - 90g = 0 \end{cases}$$

Question 14 continues on the next page

Question 14 continued

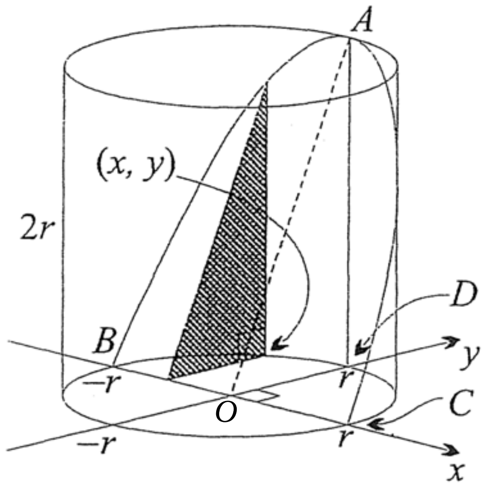
(f) Show that $\tan\theta = \dfrac{115}{25\sqrt{3}-10}$. **2**

(g) Hence, find the value of T_2 correct to the nearest integer. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a cylindrical wedge *ABCD*, the cross sections of which are all right triangles.
Each cross section is similar to triangle *AOD*.
The base of each cross section is parallel to *OD*.
The height of the cylinder is equal to the diameter of its base.
Let the radius of the base be r units.



(i) Show that the typical triangular cross-section shaded has area $\left(r^2 - x^2\right)$ square units. **2**

(ii) Hence find the volume of the wedge. **2**

(b) For positive real numbers x and y

(i) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ **2**

When is there equality?

(ii) Hence by considering $\dfrac{1}{a} + \dfrac{1}{b}$, or otherwise, prove that $\dfrac{2ab}{a+b} \leq \sqrt{ab}$ **1**

for positive real numbers a, b.

(iii) Hence, or otherwise prove that $\dfrac{1}{x-1} + \dfrac{1}{x} + \dfrac{1}{x+1} > \dfrac{3}{x}$ for any $x > 1$ **2**

(iv) If $H = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \ldots + \dfrac{1}{n}$, where n is an integer $n > 1$, **2**

use (iii) to show that $\lim\limits_{n \to \infty} H = \infty$.

Question 15 continues on the next page

(c) (i) Given that ω is one of the non-real roots of $z^3 = 1$, show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Using (i), or otherwise, show that **3**

$$\left(\frac{\omega}{1+\omega}\right)^k + \left(\frac{\omega^2}{1+\omega^2}\right)^k = (-1)^k \, 2\cos\tfrac{2}{3}k\pi \text{, where } k \in \mathbb{Z}.$$

End of Question 15

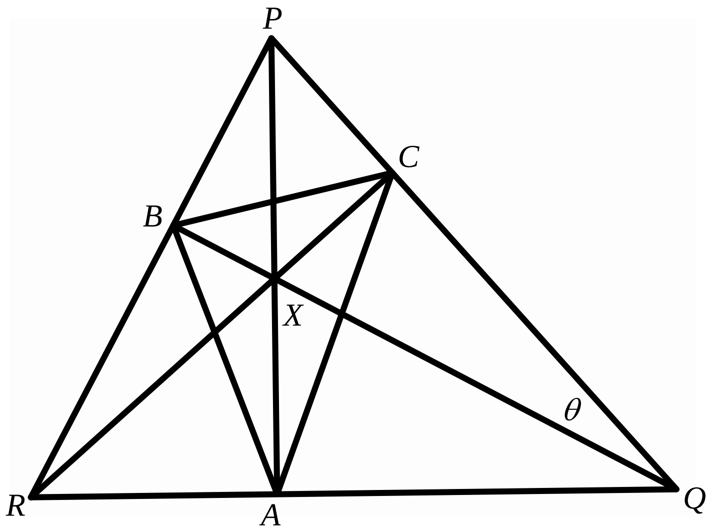
Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) $I_n = \int_0^a (a-x)^n \cos x \, dx$, $a > 0$ and n is an integer with $n \geq 0$.

(i) Show that, for $n \geq 2$, $I_n = na^{n-1} - n(n-1)I_{n-2}$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^3 \cos x \, dx$ **3**

(b) In the figure below, ΔPQR is acute angled and *AP*, *BQ* and *CR* are altitudes concurrent at *X*.
Also $\angle XQC = \theta$.
ΔABC is called the *pedal triangle* of ΔPQR.



(i) Prove that $\angle XRB = \theta$. **2**

(ii) Prove that *X*, *A*, *Q* and *C* are concyclic. **1**

(iii) Deduce that $\angle XAC = \theta$. **1**

(iv) Hence deduce that in an acute angled triangle the altitudes bisect the angles of the pedal triangle through which they pass. **2**

Question 16 continues on the next page

Question 16 continued

(c) (i) A binary string is a sequence of **1**s and **0**s,
e.g. **1 1 0 1 1 1 1 0 0 1 0 1** is a binary string of length 12.

In a binary string of length 50, how many ways are there to have a string with exactly 9 **1**s and that no two **1**s are adjacent? Justify your answer. **2**

(ii) Given 50 cards with the integers 1, 2, 3, … 50 printed on them, how many ways are there to select 9 distinct cards, such that no two cards have consecutive numbers printed on them? **1**
(An answer with no reasoning will get no credit.)

End of paper

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Student Number:________________________________

Mathematics Extension 2
Trial HSC 2012

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct ↙

A ⊗ B ⊗ C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. A B C **D**
2. A **B** C D
3. A B **C** D
4. **A** B C D
5. A B **C** D
6. **A** B C D
7. A **B** C D
8. A **B** C D
9. A B C **D**
10. **A** B C D

Q11 (a) $I = \int_0^{1/2} \frac{\sqrt{x}}{(1-x)^{3/2}}$ let $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/4} \frac{\sin\theta}{(1-\sin^2\theta)^{3/2}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/4} \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta}$$

$$= 2\int_0^{\pi/4} \tan^2\theta \, d\theta$$

$$= 2 \int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$$

$$= 2\left[\tan\theta - \theta\right]_0^{\pi/4}$$

$$= 2\left[1 - \frac{\pi}{4}\right]$$

$$= 2 - \frac{\pi}{2}$$

(b) $\int x\sqrt{3-x}\, dx$ let $u = 3-x$ $\therefore x = 3-u$, $dx = -du$

$$= -\int (3-u)\sqrt{u}\, du$$

$$= \int (u^{3/2} - 3u^{1/2})\, du$$

$$= \frac{2}{5}u^{5/2} - 2u^{3/2} + c$$

$$= \frac{2}{5}(3-x)^{5/2} - 2(3-x)^{3/2} + c.$$

Q11 (CONTD)

c (i)
$$\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{\sqrt{2x-4x^2}}\,dx.$$
$$= \frac{1}{2}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{x}{2}-x^2}}$$
$$= \frac{1}{2}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{16}-(x-\frac{1}{4})^2}}$$
$$= \frac{1}{2}\left[\sin^{-1}\frac{x-\frac{1}{4}}{\frac{1}{4}}\right]_{\frac{1}{8}}^{\frac{1}{4}}$$
$$= \frac{1}{2}\left[\sin^{-1}(4x-1)\right]_{\frac{1}{8}}^{\frac{1}{4}}$$
$$= \frac{1}{2}\left[\sin^{-1}0 - \sin^{-1}(-\tfrac{1}{2})\right]$$
$$= \frac{1}{2}\left(0 - -\frac{\pi}{6}\right)$$
$$= \frac{\pi}{12}.$$

(ii)
$$\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1-x}{\sqrt{2x-4x^2}}\,dx = \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{8-8x}{\sqrt{2x-4x^2}}\,dx$$
$$= \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{6+2-8x}{\sqrt{2x-4x^2}}\,dx.$$
$$= \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{6}{\sqrt{2x-4x^2}}\,dx + \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{2-8x}{\sqrt{2x-4x^2}}\,dx.$$
$$= \frac{3}{4}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{2x-4x^2}} + \frac{1}{4}\left[\sqrt{2x-4x^2}\right]_{\frac{1}{8}}^{\frac{1}{4}}$$
$$= \frac{3}{4}\times\frac{\pi}{12} + \frac{1}{4}\left[\sqrt{\tfrac{1}{4}} - \sqrt{3/16}\right]$$
$$= \frac{\pi}{16} + \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$$
$$= \frac{\pi}{16} + \frac{1}{8} - \frac{\sqrt{3}}{16}.$$

Q11 CONTD

(d) $\Delta = 16i^2 + 8(1-i^2)$

$= -16 + 8(2)$

$= 0.$

(e) (i) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6$

$= \cos\pi + i\sin\pi$

$= -1.$

(ii) $LHS = (\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta)$

$= \text{cis}\,\theta\,(1 + \text{cis}(-\theta))$

$= \text{cis}\,\theta + \text{cis}\,0$

$= \text{cis}\,\theta + 1$

$= 1 + \cos\theta + i\sin\theta$

$= RHS.$

(iii) $\left(1 + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= \left[\text{cis}\,\frac{\pi}{6}\left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^6 + \left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= -1\left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= 0$ as required.

2012 Extension 2 Mathematics THSC:
Solutions— Question 12

Marks

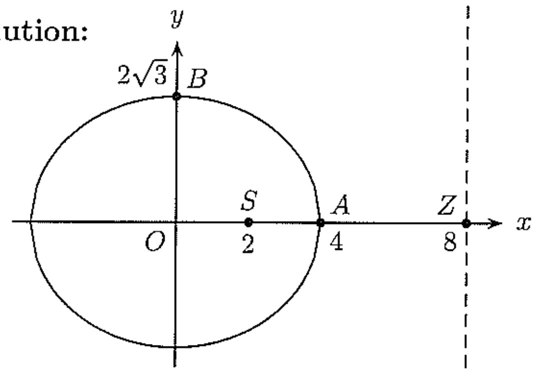
Question 12 (15 marks)

(a) The line $x = 8$ is a directrix of the ellipse with equation **2**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$$

and (2, 0) is the corresponding focus.
Find the value of a and b.

Solution:



$$\begin{aligned} ae &= 2 \quad \ldots \quad \boxed{1} \\ \frac{a}{e} &= 8 \quad \ldots \quad \boxed{2} \\ \boxed{1} \times \boxed{2}: \; a^2 &= 16, \\ \therefore a &= 4. \\ \frac{SA}{AZ} &= e, \\ &= \tfrac{2}{4}, \\ &= \tfrac{1}{2}. \\ b^2 &= a^2(1 - e^2), \\ &= 16(1 - \tfrac{1}{4}), \\ &= 12, \\ \therefore b &= 2\sqrt{3}. \end{aligned}$$

(b) (i) Show that $2 - i$ is a solution of the equation $z^3 - (2 - i)z^2 + z - 2 + i = 0$. **2**

Solution:

$$\begin{aligned} \text{L.H.S.} &= (2 - i)^3 - (2 - i)(2 - i)^2 + (2 - i) - (2 - i), \\ &= 0, \\ &= \text{R.H.S.} \end{aligned}$$

$\therefore (2 - i)$ is a solution.

(ii) Hence find all the solutions of the equation $z^3 - (2 - i)z^2 + z - 2 + i = 0$. **2**

Solution:

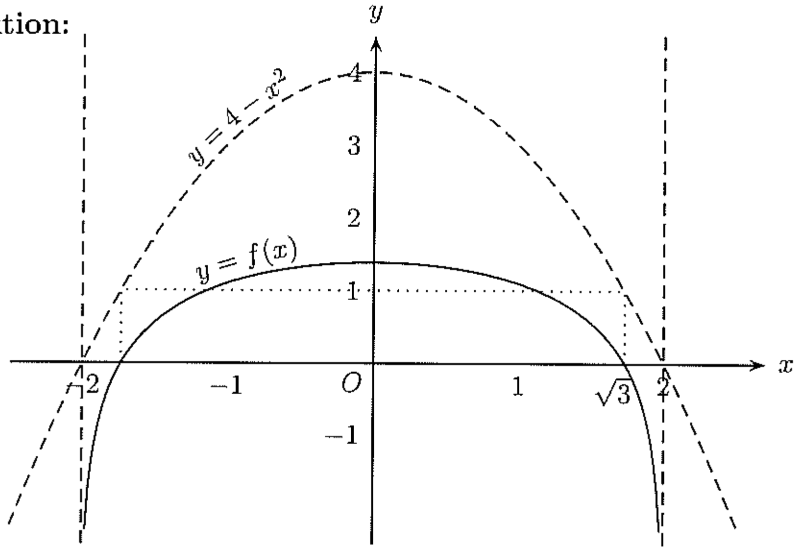
$$\begin{aligned} z(z^2 + 1) - (2 - i)(z^2 + 1) &= 0, \\ (z - 2 + i)(z^2 + 1) &= 0, \\ (z - 2 + i)(z + i)(z - i) &= 0. \end{aligned}$$

$\therefore$ Solutions $2 - i$, $\pm i$.

(c) Consider the function $f(x) = \log_e(4 - x^2)$.

(i) By first sketching $y = 4 - x^2$, sketch $y = f(x)$. [2]

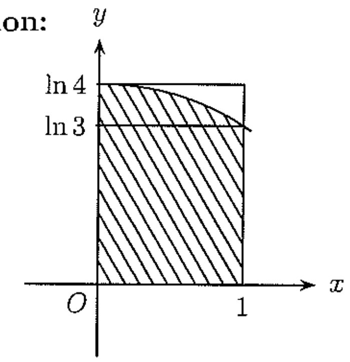
Solution:



Let A be the magnitude of the area enclosed by the graph of $y = f(x)$, the coordinate axes and the line $x = 1$.

(ii) Without evaluating A, use (i) to show that $\log_e 3 < A < \log_e 4$. [1]

Solution:



When $x = 0$, $f(0) = \ln 4$,
$x = 1$, $f(1) = \ln 3$.
From the sketch, it is clear that the shaded area A is between the upper rectangle's area $(1 \times \ln 4)$ and the lower rectangle's area $(1 \times \ln 3)$.

(iii) Find $\displaystyle\int \frac{x^2}{4-x^2}\,dx$. [3]

Solution: Method 1—

$$\begin{aligned} \text{I} &= \int \frac{x^2-4+4}{4-x^2}\,dx, \\ &= -\int dx + \int \frac{4}{4-x^2}\,dx, \\ &= -x + \int \frac{dx}{2-x} + \int \frac{dx}{2+x}, \\ &= -x - \ln(2-x) + \ln(2+x) + c, \\ &= \ln\left(\frac{2+x}{2-x}\right) - x + c. \end{aligned}$$

$$\begin{aligned} \frac{4}{4-x^2} &\equiv \frac{A}{2-x} + \frac{B}{2+x}, \\ 4 &\equiv A(2+x) + B(2-x), \\ \text{put } x &= -2, \quad B = 1, \\ x &= 2, \quad A = 1. \end{aligned}$$

Solution: Method 2—

$$\begin{aligned} \text{I} &= \int \frac{4\sin^2\theta.2\cos\theta\,d\theta}{4-4\sin^2\theta}, \\ &= \int \frac{2\sin^2\theta.\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}, \\ &= 2\int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta, \\ &= 2\int(\sec\theta - \cos\theta)\,d\theta, \\ &= 2\big\{\ln(\sec\theta+\tan\theta) - \sin\theta\big\} + c, \\ &= 2\ln\left(\frac{2}{\sqrt{4-x^2}} + \frac{x}{\cancel{2}}.\frac{\cancel{2}}{\sqrt{4-x^2}}\right) - x + c, \\ &= 2\ln\left(\frac{x+2}{\sqrt{(2+x)(2-x)}}\right) - x + c, \\ &= \ln\left(\frac{2+x}{2-x}\right) - x + c. \end{aligned}$$

$$\begin{aligned} \text{Put } x &= 2\sin\theta, \\ dx &= 2\cos\theta d\theta, \\ \frac{x^2}{4} &= \sin^2\theta, \\ \frac{4-x^2}{4} &= \cos^2\theta. \end{aligned}$$

(iv) Hence find the exact value of A in the form $a + b\log_e c$, where a, b and c are integers. [3]

Solution:

$$\begin{aligned} \text{I} &= \int_0^1 \ln(4-x^2)\,dx, \\ &= x\ln(4-x^2)\Big]_0^1 + 2\int_0^1 \frac{x^2\,dx}{4-x^2}, \\ &= \{\ln 3 - 0\} + 2\left[\ln\left(\frac{2+x}{2-x}\right) - x\right]_0^1, \\ &= \ln 3 + 2\{\ln 3 - 1 - (\ln 1 - 0)\}, \\ &= 3\ln 3 - 2. \end{aligned}$$

$$\begin{aligned} u &= \ln(4-x^2), \quad v' = 1, \\ u' &= \frac{-2x}{4-x^2}, \quad v = x. \end{aligned}$$

$\overline{15}$ Ext 2

13. (a) Prove $n^{n+1} > n(n+1)^{n-1}$ for $n \geq 2$.

Let $n = 2$. $LHS = 2^3 = 8$, $RHS = 2(3)^1 = 6$

$\therefore$ true for $n = 2$.

Assume true for $n = k$

$$k^{k+1} > k(k+1)^{k-1}$$

or $k^k > (k+1)^{k-1}$ $*$

or $\dfrac{k^k}{(k+1)^{k-1}} > 1$.

Let $n = k+1$ RTP $(k+1)^{k+1} > (k+2)^k$

OR $\dfrac{(k+1)^{k+1}}{(k+2)^k} > 1$

Now $LHS = \dfrac{(k+1)^{k+1}}{(k+2)^k}$

$$> \frac{(k+1)^{k+1}}{(k+2)^k} \times \frac{(k+1)^{k-1}}{k^k}$$ since $\dfrac{(k+1)^{k-1}}{k^k} < 1$ from assumption

$$> \frac{(k+1)^{2k}}{[k(k+2)]^k}$$

$$= \frac{[(k+1)^2]^k}{[k(k+2)]^k}$$

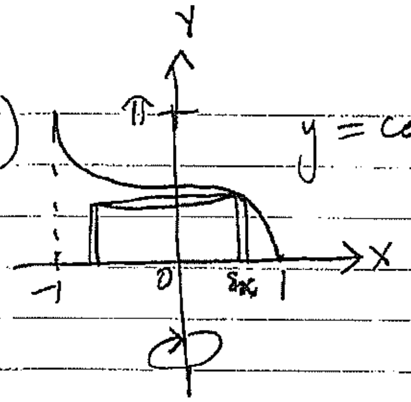
$$= \left[\frac{k^2+2k+1}{k^2+2k}\right]^k > 1 \quad \#$$

$\therefore$ true for $n = k+1$.

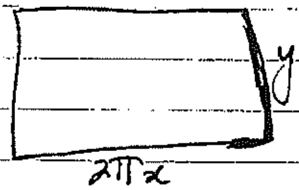
$\therefore$ By POMI, true $\forall n \geq 2$.

(3)

13 (b)



Area of cylindrical slice $= 2\pi x y$



Vol. of cyl. slice $= 2\pi x y \, \delta x$.

$$V = \lim_{\delta x \to 0} \sum_{x=-1}^{1} 2\pi x y \, \delta x.$$

$$= 2\pi \int_{-1}^{1} x \cos^{-1} x \, dx$$ ✓

$$V = 4\pi \int_0^1 x . \cos^{-1} x . dx \quad (\text{symmetric})$$

$u = \cos^{-1} x \qquad dv = x$

$du = \frac{-1}{\sqrt{1-x^2}} \qquad v = \frac{x^2}{2}$.

$$= 4\pi \left[\left[\frac{x^2}{2} . \cos^{-1} x \right]_0^1 - \int_0^1 \frac{x^2}{2} \left(\frac{-1}{\sqrt{1-x^2}} \right) dx \right]$$ ✓

$$= 2\pi \left[x^2 \cos^{-1} x \right]_0^1 + 2\pi \int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx$$

$$= 2\pi \left[x^2 \cos^{-1} x \right]_0^1 + 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta$$

Let $x = \sin \theta$

$dx = \cos \theta \, d\theta$.

$x = 0, \ \theta = 0$

$x = 1, \ \theta = \frac{\pi}{2}$

$$= 2\pi \left[x^2 \cos^{-1} x \right]_0^1 + 2\pi . \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2\theta) \, d\theta$$

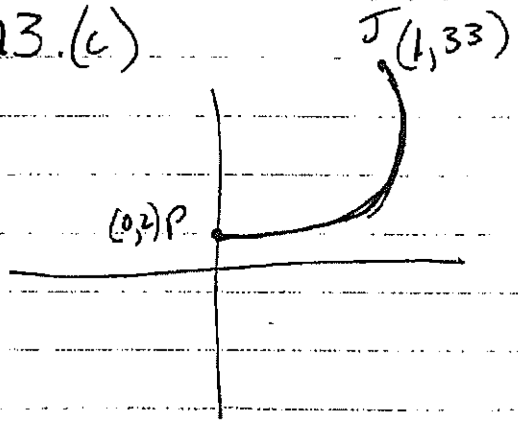
$$= 0 + \pi \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

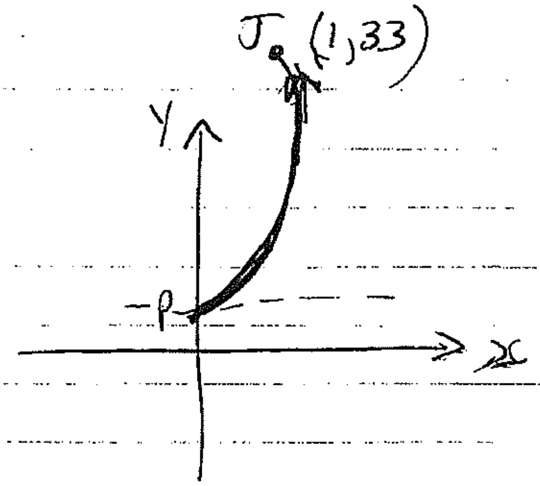
$$= \pi \left(\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right)$$

$$= \frac{\pi^2}{2} \text{ cubic units.}$$ ✓

(3)

13.(c)





(i) $x = 2\sin\frac{2t}{15}$ $\quad y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

$0 \le t \le \frac{15\pi}{2}$

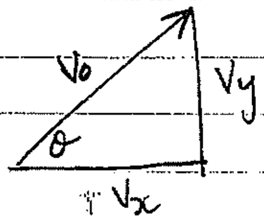
At $t=0$, $x=0$, $y=2$ ✓ ①

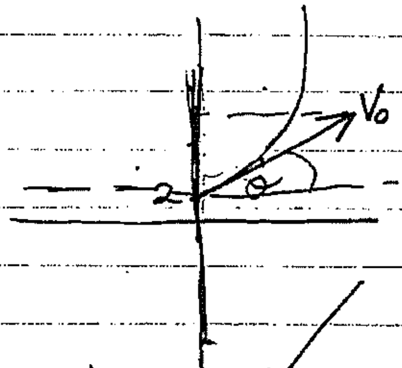
$\Rightarrow \underline{P = (0,2)}$

(ii) $V_x = 2\cos\frac{2t}{15} \times \frac{2}{15}$ and $\underline{V_y = \frac{5}{3} - \frac{5}{9}\cos\frac{t}{3}}$ ② ✓

$\underline{V_x = \frac{4}{15}\cos\frac{2t}{15}}$ ① ✓ ②

(iii)





Velocity components

$V_x = V\cos\alpha$

$V_y = V\sin\alpha$.

At $t=0$, $\underline{V_x = \frac{4}{15}}$, $\underline{V_y = \frac{10}{9}}$ ✓

②

Then $\tan\theta = \frac{V_y}{V_x}$

$= \frac{10}{9} \times \frac{15}{4}$

$\tan\theta = \frac{25}{6}$

$\theta = 1.33$ radians

or 76.5°

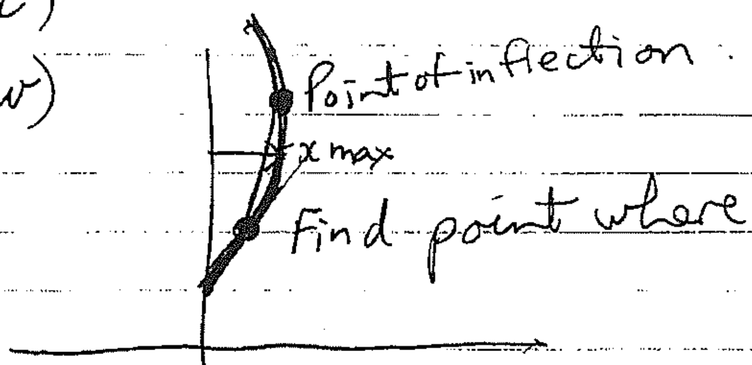
Then angle made with forward direction

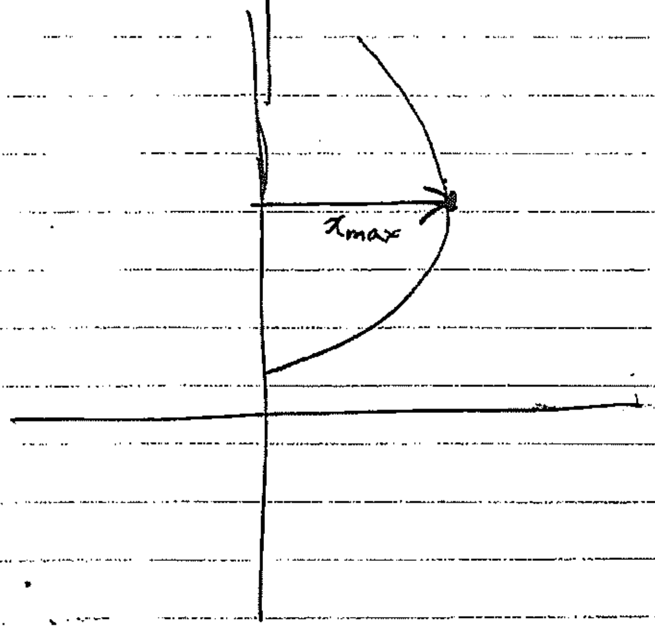
$= 90 - 76.5$

$= 13.5^\circ$ ✓

13(c)(iv)

*





$x = 2\sin\left(\frac{2t}{15}\right)$

$x' = \frac{4}{15}\cos\frac{2t}{15} = 0$ for stat pts

$\Rightarrow \cos\frac{2t}{15} = 0$ ✓ $\boxed{\frac{1}{2}}$

$\frac{2t}{15} = \frac{\pi}{2}, \frac{3\pi}{2}$ ✓ $\boxed{\frac{1}{2}}$

$t = \frac{15\pi}{4}, \frac{45\pi}{4}$ ← too big

$t = 11.78$ seconds

∴ at $t = 11.8$ seconds the ball starts to swing towards the jack. ✓ $\boxed{\frac{1}{2}}$

(2)

$0 < t < \frac{15\pi}{2}$

13(c)

(v) $x = 2\sin\frac{2t}{15}$, $y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

Find t when $x = 1$, $y = 33$

$\Rightarrow 1 = 2\sin\frac{2t}{15}$ (1) and $33 = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$ (2)

$\sin\frac{2t}{15} = \frac{1}{2}$ $\qquad 31 = \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

$\frac{2t}{15} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$

$t = \frac{15\pi}{12}, \frac{75\pi}{12}, \frac{195\pi}{12}$ too big.

$t = \frac{5\pi}{4}, \frac{25\pi}{4}$,

Determine if values $t = \frac{15\pi}{12}$ or $\frac{75\pi}{12}$ satisfy (2).

$t = \frac{15\pi}{12} \Rightarrow 31 = \frac{5}{3} \times \frac{15\pi}{12} - \frac{5}{3}\sin\left(\frac{15\pi}{36}\right) = 4.935$ ×

$t = \frac{75\pi}{12} \Rightarrow 31 = \frac{5}{3} \times \frac{75\pi}{12} - \frac{5}{3}\sin\left(\frac{75\pi}{12}\right)$

$31 = 31.546$ × . (cloze!)

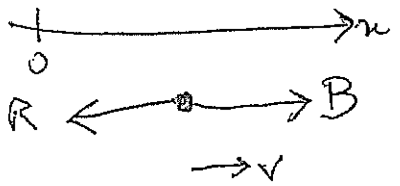
∴ Ball does not hit the Jack.

(2)

EXT2 THSG 2012

Question 14

(a)



Nett Force = B − R

$= 410 - 50$

$= 360\text{ N}$

$F = ma$

$\therefore 360 = 90a$

$a = 4\text{ m/s}^2$ [2]

(b) Given $a = 4$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4$

Integrate w.r.t. x:

$\frac{1}{2}v^2 = 4x + c$

$v^2 = 2x + c'$

Let $x = 0$ when $t = 0, v = 0$

$\therefore c' = 0$

$\therefore v^2 = 8x$

When $x = 32$

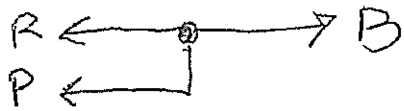
$v^2 = 8 \times 32$

$= 256$

$\therefore v = 16\text{ m/s}$ $(v > 0)$

[2]

(c)



$P = 6v\text{ N}$

$F = ma$

$410 - 50 - 6v = 90a$

$a = \frac{360 - 6v}{90}$

[1] $\therefore a = \frac{1}{15}(60 - v)$

(d) $\frac{dv}{dt} = \frac{1}{15}(60 - v)$

$\frac{dt}{dv} = \frac{15}{60 - v}$

Time from $v = 16$ to $v = 20$

$t = \int_{16}^{20} \frac{15}{60 - v}\, dv$

$= 15\left[-\ln(60 - v)\right]_{16}^{20}$

$= -15\left[\ln 40 - \ln 44\right]$

$\doteqdot 1.4\text{ s}$

[3]

EXT 2 THSC 2012

Q 14 (cont'd)

(e) Horizontal $a=0$ $\longleftrightarrow +$

Forces Boat: $500\cos 30^\circ$

Resistance: -100

Parachute: $-T_2\cos\theta$

$\therefore 500\cos 30^\circ - T_2\cos\theta - 100 = 0$ —(1)

Vertical $a=0$ $\updownarrow +$

Forces Boat: $-500\sin 30^\circ$

Weight: $-90g$

Parachute: $T_2\sin\theta$

$\therefore T_2\sin\theta - 500\sin 30^\circ - 90g = 0$ —(2)

[3]

(f) From (1)

$T_2\cos\theta = 500\cos 30^\circ - 100$

$= 250\sqrt{3} - 100$

From (2)

$T_2\sin\theta = 500\sin 30^\circ + 90g$

$= 500\times\frac{1}{2} + 900$

$= 1150$

$\therefore \tan\theta = \dfrac{1150}{250\sqrt{3}-100}$

$= \dfrac{115}{25\sqrt{3}-10}$

[2]

(g) $T_2 = \dfrac{1150}{\sin\theta}$

from (f) $\theta \doteqdot 73.85^\circ$

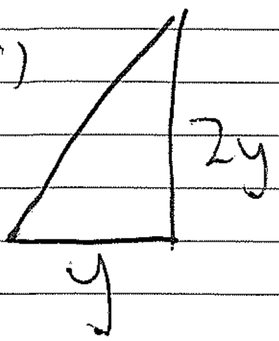
$\therefore T_2 = 1197$ N

[2]

Q15

(a)

(i)



$$\text{Area} = \frac{1}{2} \times 2y \times y$$

$$= y^2.$$

Since $x^2 + y^2 = r^2$

$$\text{Area} = r^2 - x^2$$

2

(ii)

$$\delta V = (r^2 - x^2)\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=-r}^{r} (r^2 - x^2)\,\delta x.$$

$$= \int_{-r}^{r} r^2 - x^2 \, dx.$$

$$= 2\int_{0}^{r} r^2 - x^2 \, dx$$

$$= 2\left[xr^2 - \frac{x^3}{3}\right]_0^r$$

$$= 2\left(r^3 - \frac{r^3}{3}\right)$$

$$= \frac{4r^3}{3}.$$

2

(b) (i) $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$.

$\frac{a^2+b^2}{2} \geq ab$.

let $x = a^2 \Rightarrow a = \sqrt{x}$
$y = b^2 \Rightarrow b = \sqrt{y}$.

$\frac{x+y}{2} \geq \sqrt{xy}$.

2

(ii) $\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \frac{1}{\sqrt{ab}}$

$\frac{a+b}{2ab} \geq \frac{1}{\sqrt{ab}}$.

1

$\frac{2ab}{a+b} \leq \sqrt{ab}$.

(iii) $\frac{x(x+1) + x^2-1 + x(x-1)}{x(x^2-1)} = \frac{3x^2-1}{x(x^2-1)}$

$= \frac{3x^2-3+2}{x(x^2-1)}$

$= \frac{3(x^2-1)}{x(x^2-1)} + \frac{2}{x(x^2-1)}$

$$= \frac{3}{x} + \frac{2}{x(x^2-1)}$$

$x^2 - 1 > 0$ since $x > 1$

$\therefore x(x^2-1) > 0$

$\therefore \frac{2}{x(x^2-1)} > 0$

So $\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$.

(iv)

$$H = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \dots + \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right)$$

$$= 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \dots + \frac{3}{n-1} \quad \text{let } n-1 = 3k.$$

$$= 1 + 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \dots + \frac{1}{k}.$$

$$= 1 + 1 + 1 + \dots + \frac{3}{k-1} \quad \text{let } k-1 = 3m.$$

$$= 1 + 1 + 1 + \dots + \frac{1}{m}.$$

As $n \to \infty$ this process can be continued

Thus $H = 1 + 1 + 1 + 1 + \dots + 1 + \dots$

$\therefore \lim_{n \to \infty} H < \infty$

(c) (i) Since w is a solution of

$z^3 = 1$

$\therefore w^3 - 1 = 0.$

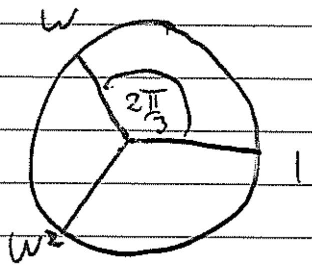
$(w-1)(w^2+w+1) = 0$

Since w is a non-real root

$w \neq 1.$

$\therefore w^2 + w + 1 = 0.$

(ii)



$\arg(w) = \frac{2\pi}{3}$

Given that $1 + w + w^2 = 0$

so $1 + w = -w^2$

$1 + w^2 = -w$

$LHS = \left(\frac{w}{-w^2}\right)^k + \left(\frac{w^2}{-w}\right)^k$

$= (-1)^k (w^{-k}) + (-1)^k (w^k)$

$= (-1)^k (w^{-k} + w^k)$

Given $z^n + z^{-n} = 2\cos n\theta$

$= (-1)^k 2\cos\left(\frac{2\pi}{3}k\right)$

Q16.

(a) (i) $$I_n = \int_0^a (a-x)^n \cos x \, dx \quad \text{- - - -} \quad (A)$$

$$= \int_0^a (a-x)^n \frac{d}{dx}(\sin x) \, dx$$

$$= \Big[(a-x)^n \sin x\Big]_0^a - n\int_0^a (a-x)^{n-1} \sin x \, dx$$

$$= 0 - n\int_0^a (a-x)^{n-1} \sin x \, dx$$

$$= -n\int_0^a (a-x)^{n-1} \frac{d}{dx}(-\cos x) \, dx$$

$$= \Big[-n\big((a-x)^{n-1} \cdot -\cos x\big)\Big]_0^a + n(n-1)\int_0^a (a-x)^{n-2} \cdot -\cos x \, dx$$

$$= \big(0 + n a^{n-1}\big) - n(n-1)\int_0^a (a-x)^{n-2} \cos x \, dx$$

$$\therefore \boxed{I_n = n a^{n-1} - n(n-1) I_{n-2}} \quad (B) \quad \boxed{\text{NB } n \geq 2}$$

(ii) $$I_3 = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^3 \cos x \, dx$$

$$= 3\left(\frac{\pi}{2}\right)^2 - 6 I_1 \quad \text{—} \quad (C)$$

now $$I_1 = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \cos x \, dx \quad \text{from } (A)$$

[Don't use (B)]

$$= \int_0^{\pi/2} \frac{\pi}{2} \cos x \, dx - \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= \Big[\frac{\pi}{2} \sin x\Big]_0^{\frac{\pi}{2}} - \Big[\big[x \sin x\big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx\Big]$$

$$= \frac{\pi}{2} - \Big[\frac{\pi}{2} - 0 - \big[\cos x\big]_0^{\frac{\pi}{2}}\Big]$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right)$$

$$= 1$$

$$\therefore \boxed{I_3 = \frac{3\pi^2}{4} - 6} \quad \text{from } (C)$$

Q16. (CONTD)

(b) (i) BCQR is a cyclic quadrilateral

$\angle RCQ = \angle RBQ$ (90° altitudes)

ie angles are equal subtended by the chord RQ.

∴ ∴ $\angle BRX = \angle BQC$ (angles in the same segment are equal)

ie $\angle XRB = \theta$.

(ii) $\angle XCQ = \angle XAQ = 90^\circ$ (data)

∴ XAQ & C are concyclic (opposite angles supplementary)

(iii) $\angle XQC = \angle XAC = \theta$ (angles in the same segment standing on the same arc are equal.)

(iv) BXAR is a cyclic quadrilateral

$\angle XBR = \angle XAR = 90^\circ$ (data)

(opposite angles supplementary)

∴ $\angle XRB = \angle XAB = \theta$ (angles in the same segment standing on same arc are equal)

∴ $\angle XAB = \angle XAC = \theta$

∴ altitude PA bisects $\angle BAC$.

Similarly for the other angles.

Q16 (CONTD)

(C) (i) Place the 41 0's

choose the gaps and the end places to places the 1's. (there are 42 spaces)

ie. $\binom{42}{9}$ ways to place the 1's.

(ii) Same logic as the above.

ie place the nine consecutive cards into the 42 spaces such that no two consecutive cards are adjacent

ie $\binom{42}{9}$.